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Analyticity of Smooth Eigenfunctions and Spectral Analysis of the Gauss Map

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We provide a sufficient condition of analyticity of infinitely differentiable eigenfunctions of operators of the form $Uf(x) = \int a(x, y)\, f(b(x, y))\, \mu(dy)$ acting on functions $f: [u, v] \to \mathbb{C}$ (evolution operators of one-dimensional dynamical systems and Markov processes have this form). We estimate from below the region of analyticity of the eigenfunctions and apply these results for studying the spectral properties of the Frobenius–Perron operator of the continuous fraction Gauss map. We prove that any infinitely differentiable eigenfunction f of this Frobenius–Perron operator, corresponding to a non-zero eigenvalue admits a (unique) analytic extension to the set $\mathbb{C} \setminus (-\infty, -1]$. Analyzing the spectrum of the Frobenius–Perron operator in spaces of smooth functions, we extend significantly the domain of validity of the Mayer and Röpstorff asymptotic formula for the decay of correlations of the Gauss map.

KEY WORDS: Gauss map; Frobenius–Perron operators; analytic extension; decay of correlations; spectral decomposition.

1. INTRODUCTION

The *Gauss* or *continuous fractions* map

$$G: (0, 1) \to [0, 1), \qquad G(x) = 1/x \pmod 1 \tag{1}$$

is one of the most interesting exact dynamical systems with origin not only in number theory[(1–4)] but also in cosmology since G is an approximation of the Poincare return map of the Mixmaster cosmological model. For the

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derivation of Mixmaster Universe model and Poincare return map from Einstein equations we refer to refs. 5–8 and references therein. The density of the unique absolutely continuous invariant Borel probability measure of the Gauss map is $\rho(x) = 1/[(1+x)\ln 2]$. The Frobenius–Perron operator[9] of the Gauss map (1) with respect to this measure is[2]

$$U_G f(x) = \sum_{n=1}^{\infty} a_n(x)\, f\,(b_n(x)) \tag{2}$$

where

$$a_n(x) = \frac{x+1}{(x+n)(x+n+1)} \qquad \text{and} \qquad b_n(x) = \frac{1}{x+n}. \tag{3}$$

It is well-known[1, 9] that the spectrum of the Frobenius–Perron operator of an exact endomorphism S in $L_2(\mu)$ (μ is the absolutely continuous probability invariant measure of S) is the closed unit disk and that any point $z \in \mathbb{C}$ with $|z| < 1$ is an eigenvalue of infinite multiplicity. Nevertheless the spectral analysis of Frobenius–Perron operators is a powerful tool for studying unstable dynamics[9] because the spectra of Perron–Frobenius operators in some natural function spaces (like smooth or analytic functions) are often countable and consist of isolated eigenvalues of finite multiplicity. These eigenvalues are also known as resonances and determine the decay of the correlation functions. For piecewise analytic expanding maps one can apply the the dynamical zeta-function method[10–18] to estimate the resonances. Moreover, for some important examples of expanding maps one can find the resonances and corresponding eigenfunctions explicitly and obtain a spectral decomposition formula, representing the action of the Frobenius–Perron operator on a certain function space.[19–24]

For the Gauss map Mayer and Röpstorff gave[2] (see also refs. 3 and 4, Chapter 7) some estimations of the behaviour of the Frobenius–Perron operator U_G, which we present as Theorem MR. For an operator A on a topological vector space, whose spectrum $\sigma(A)$ is a sequence of eigenvalues (of finite multiplicity) converging to 0,[25] we denote by $\lambda_n = \lambda_n(A)$, $n = 0, 1,\ldots$ the eigenvalues of A, enumerated (taking into account the multiplicity) in such a way that $|\lambda_{n+1}| \leqslant |\lambda_n|$ for all $n \in \mathbb{N}$ and $\arg \lambda_n \leqslant \arg \lambda_{n+1}$ if $|\lambda_{n+1}| = |\lambda_n|$.

Theorem MR. Let $\mathscr{H}$ be the space of functions, holomorphic in the half-plane $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z > -\frac{1}{2}\}$, bounded in $\{z \in \mathbb{C} : \operatorname{Re} z > -\frac{1}{2} + \varepsilon\}$ for any $\varepsilon > 0$ and square-integrable with the density

$$\varkappa(x+iy) = \begin{cases} \pi/[(y^2+(1+x)^2)] & \text{if } \ 0 < x < 1/2, \\ 0 & \text{if } \ x \notin (0, 1/2). \end{cases}$$

Then $\mathscr{H}$ is a separable Hilbert space with the scalar product

$$\langle g, f\rangle_{\mathscr{H}} = \iint_{\Omega} f(x+iy)\,\overline{g(x+iy)}\,\varkappa(x+iy)\,dx\,dy,$$

$U_G(\mathscr{H}) \subseteq \mathscr{H}$, where U_G is the Frobenius–Perron operator (2), the operator $U_{\mathscr{H}} = U_G|_{\mathscr{H}}: \mathscr{H} \to \mathscr{H}$ is nuclear and self-adjoint, $\ker U_{\mathscr{H}} = \{0\}$, $\lambda_0(U_{\mathscr{H}}) = 1$ and

$$-0.30366327 \leqslant \lambda_1(U_{\mathscr{H}}) \leqslant -0.30366299, \qquad 0.10088 \leqslant \lambda_2(U_{\mathscr{H}}) \leqslant 0.10094. \tag{4}$$

Moreover, for any $f \in \mathscr{H}$ and any $g \in L_2[0, 1]$,

$$\langle g, U_G^n f\rangle = C + O(q^n), \qquad \text{where} \quad C = \int_0^1 \frac{f(x)}{\ln 2(1+x)}\,dx \cdot \int_0^1 g(y)\,dy, \tag{5}$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $L_2[0, 1]$ and $q = |\lambda_1(U_{\mathscr{H}})|$.

Although Theorem MR gives us information about the resonances of the Frobenius–Perron operator (2) in the space $\mathscr{H}$, it is not clear whether the eigenvalues and corresponding eigenspaces of U_G in $\mathscr{H}$ coincide with the eigenvalues and eigenspaces in natural larger spaces: the space of analytic functions on $[0, 1]$ and the Fréchet space $C^\infty[0, 1]$. It is also not clear whether eigenfunctions in $\mathscr{H}$ admit analytic extension outside the half-plane $\{z: \operatorname{Re} z > -1/2\}$. In this paper we clarify both points. Namely, we prove that non-zero eigenvalues and eigenspaces of U_G in the space $C^\infty[0, 1]$ coincide with the eigenvalues and corresponding eigenspaces of U_G in $\mathscr{H}$. We also prove that any infinitely differentiable eigenfunction of U_G corresponding to a non-zero eigenvalue admits a holomorphic extension to $\mathbb{C}\backslash(-\infty, -1]$ (Theorem 2). Based on these results, we extend the domain of the validity of the asymptotic formula (5) for the decay of correlation functions (Theorem 3). The proof of Theorems 2 and 3 is based on Theorem 1, which gives a condition of analyticity of smooth eigenfunctions of integral operators of the form

$$Uf(x) = \int_Y a(x, y)\, f(b(x, y))\,\mu(dy) \tag{6}$$

acting on functions $f: [u, v] \to \mathbb{C}$.

2. FORMULATION OF MAIN RESULTS

2.1. Analyticity of Smooth Eigenfunctions of Operators (6)

Let μ be a σ-additive positive finite measure on the measurable space $(Y, \mathscr{F})$, $u, v \in \mathbb{R}$ and $u < v$. By $\mathscr{A}$ we denote the space of measurable maps $\varphi: [u, v] \times Y \to \mathbb{C}$ for which there exists a neighborhood O of $[u, v]$ in $\mathbb{C}$ and a bounded measurable function $\tilde{\varphi}: O \times Y \to \mathbb{C}$ such that $\tilde{\varphi}$ is holomorphic with respect to the first variable and $\tilde{\varphi}(x, y) = \varphi(x, y)$ for all $(x, y) \in [u, v] \times Y$. For $f: [u, v] \to \mathbb{C}$ and $\varphi: [u, v] \times Y \to \mathbb{C}$ infinitely differentiable with respect to the variable $x \in [u, v]$ we denote

$$
\begin{aligned}
M_n(f) &= \frac{1}{n!} \max_{x \in [u, v]} |f^{(n)}(x)|, \\
M_n(\varphi) &= \frac{1}{n!} \sup_{(x, y) \in [u, v] \times Y} \left| \frac{\partial^n \varphi}{\partial x^n}(x, y) \right|, \\
\frac{1}{r(f)} &= \overline{\lim_{n \to \infty}} (M_n(f))^{\frac{1}{n}}, \\
\frac{1}{r(\varphi)} &= \overline{\lim_{n \to \infty}} (M_n(\varphi))^{1/n}, \qquad \frac{1}{\tilde{r}(\varphi)} = \sup_{n \geqslant 2} \left(\frac{M_n(\varphi)}{M_1(\varphi)} \right)^{\frac{1}{n-1}}
\end{aligned}
\tag{7}
$$

(if $M_1(\varphi) = 0$ we put $\tilde{r}(\varphi) = +\infty$). It is worth noticing that $f: [u, v] \to \mathbb{C}$ is analytic if and only if $r(f) > 0$ and in this case $r(f)$ is precisely the maximal $\varepsilon > 0$ for which f admits a holomorphic extension to the ε-neighborhood of $[u, v]$ in $\mathbb{C}$.[28]

Let $a, b \in \mathscr{A}$ and $b([u, v] \times Y) \subset [u, v]$. We consider the operator U defined by (6) and denote

$$
\begin{aligned}
\tau = \tau(U) &= \sup_{(x, y) \in [u, v] \times Y} \left| \frac{\partial b}{\partial x}(x, y) \right|, \\
\gamma(U) &= \min\{r(a), (1 - \tau(U))\, \tilde{r}(b)\}.
\end{aligned}
\tag{8}
$$

Note that the powers U^n have the same shape (with Y^n equipped with the measure $\mu \times \cdots \times \mu$ instead of Y). This allows us to define

$$
\gamma^*(U) = \sup_{n \in \mathbb{N}} \gamma(U^n). \tag{9}
$$

Theorem 1. Let U be the operator (6) with $\gamma^*(U) > 0$, $k \in \mathbb{N}$, $z \in \mathbb{C} \setminus \{0\}$ and $f \in C^\infty[u, v]$ be such that $(U - zI)^k f = 0$. Then f is analytic in $[u, v]$ and $r(f) \geqslant \gamma^*(U)$.

2.2. Spectral Properties of the Gauss Map

For $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{N}^m$ we denote

$$b_{\mathbf{n}}(x) = b_{n_1, \ldots, n_m}(x) = b_{n_1}(b_{n_2}(\ldots(b_{n_m}(x))\ldots)), \tag{10}$$

$$a_{\mathbf{n}}(x) = a_{n_m}(x) \cdot a_{n_{m-1}}(b_{n_m}(x)) \cdot a_{n_{m-2}}(b_{n_{m-1}, n_m}(x)) \cdot \cdots \cdot a_{n_1}(b_{n_2, \ldots, n_m}(x)). \tag{11}$$

Formula (2) implies that for any $m \in \mathbb{N}$,

$$U_G^m f(x) = \sum_{\mathbf{n} \in \mathbb{N}^m} a_{\mathbf{n}}(x)\, f(b_{\mathbf{n}}(x)), \tag{12}$$

where U_G is the Frobenius–Perron operator (2) of the Gauss map (1). Let

$$U_k = (U_G)|_{C^k[0,1]} \colon C^k[0,1] \to C^k[0,1] \quad \text{and}$$
$$R_{m,k} = \sup_{x \in [0,1]} \sum_{\mathbf{n} \in \mathbb{N}^m} a_{\mathbf{n}}(x)\, |b'_{\mathbf{n}}(x)|^k. \tag{13}$$

The following Proposition 1 is the result of application of the Ruelle's theorem[13] (presented also as Theorem 2.5 in ref. 10) to the Gauss map.

Proposition 1. For any $k \in \mathbb{N}$, the essential spectral radius of the operator U_k is

$$R_k = \lim_{m \to \infty} (R_{m,k})^{1/m} = \inf_{m \in \mathbb{N}} (R_{m,k})^{1/m}, \tag{14}$$

where $R_{m,k}$ are the numbers defined by (13).

In the following theorem we use the notation of Proposition 1.

Theorem 2. Let U_G be the operator (2), $c = 4/(\sqrt{5}+1)^2$, $k \in \mathbb{N}$ and $\mathbf{H}$ be the space of all complex-valued functions, holomorphic on $\mathbb{C} \setminus (-\infty, -1]$ and bounded on each set

$$D_\varepsilon = \{z \in \mathbb{C} : |\mathrm{Im}\, z| > \varepsilon \text{ or } \mathrm{Re}\, z > -1+\varepsilon\}, \qquad \varepsilon > 0. \tag{15}$$

Then $R_k \leqslant c^k$, the set $S_k = \{z \in \sigma(U_k) : |z| > R_k\}$ is a finite set of eigenvalues of finite multiplicity and for any $z \in S_k$, the eigenspace

$$\mathscr{E}(z) = \{f \in C^k[0,1] \mid \exists n \in \mathbb{N} : (U_k - zI)^n f = 0\}$$

has the following property:

$$\mathscr{E}(z) = \{f \in C^k[0,1] \mid U_k f = zf\} \subset \mathbf{H} \subset \mathscr{H} \subset C^\infty(\mathbb{R}). \tag{16}$$

The equality from (16) implies that the restriction $(U_G)|_{\mathscr{E}(z)}$ is the scalar operator zI (no Jordan blocks appear). Formula (16) implies also that non-zero eigenvalues and corresponding eigenspaces of the operator U_G in spaces C^∞, $\mathscr{H}$ and $\mathbf{H}$ coincide.

2.3. Decay of Correlation Functions of the Gauss Map

Theorem 2 allows us to extend significantly the domain of validity of the Mayer and Roepstorff asymptotic formula (5). Below λ_n stand for $\lambda_n(U_G|_{C^\infty})$ which are equal according to Theorem 2 to $\lambda_n(U_G|_{\mathscr{H}}) = \lambda_n(U_G|_{\mathbf{H}}) = \lambda_n(U_G|_{C^\infty})$.

Theorem 3. Let U_G be the operator (2), $k \in \mathbb{N}$, $f \in C^k[0,1]$, g be any linear continuous functional on $C^k[0,1]$, $q = |\lambda_1|$ if $k \geqslant 2$ and $q \in (c,1)$ if $k=1$, where $c = 4/(\sqrt{5}+1)^2$. Then

$$\langle U_G^n f, g\rangle = C + O(q^n), \qquad \text{where} \quad C = \int_0^1 \frac{f(x)}{\ln 2(1+x)}\, dx \langle 1, g\rangle.$$

In particular, the asymptotic formula (5) is valid for any $f \in C^2[0,1]$ (not only for $f \in \mathscr{H}$).

2.4. Spectral Decomposition

The following proposition is a consequence of Theorem MR and the Hilbert–Schmidt theorem.(26)

Proposition 2. There exists an orthonormal basis f_n, $n = 0, 1, \ldots$ in the Hilbert space $\mathscr{H}$ such that for any $f \in \mathscr{H}$,

$$U_G f = \sum_{n=0}^{\infty} \lambda_n \langle f_n, f\rangle_{\mathscr{H}}\, f_n, \tag{17}$$

where the series (17) converges in the topology of the Hilbert space $\mathscr{H}$ (and therefore uniformly).

3. PROOF OF THEOREM 1

Lemma 1.1. Let $f: [u, v] \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{C}$ be two functions of class C^n. Then for any $x \in [u, v]$,

$$(g \circ f)^{(n)}(x) = n! \sum_{k_1 + \cdots + nk_n = n} \frac{g^{(k_1 + \cdots + k_n)}(f(x))}{k_1! \cdots k_n! (1!)^{k_1} \cdots (n!)^{k_n}} (f'(x))^{k_1} \cdots (f^{(n)}(x))^{k_n}, \tag{18}$$

where k_j are non-negative integers.

Proof. This formula is known as a Vallée–Poussin equality (see, e.g., ref. 27). ▮

Lemma 1.2. Let $c \in \mathbb{C}$ and $n \in \mathbb{N}$. Then

$$\sum_{k_1 + \cdots + nk_n = n} \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!} c^{k_1 + \cdots + k_n} = c(c+1)^{n-1}, \tag{19}$$

where k_j are non-negative integers.

Proof. Consider the functions $f(x) = 1/(2-x)$ and $g(x) = 1/(c+1-cx)$. Then $f^{(k)}(x) = k!/(2-x)^{k+1}$ and $g^{(k)}(x) = c^k k!/(c+1-cx)^{k+1}$. According to (18)

$$(g \circ f)^{(n)}(1) = n! \sum_{k_1 + \cdots + nk_n = n} \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!} c^{k_1 + \cdots + k_n}. \tag{20}$$

From the other side $(g \circ f)(x) = \frac{1}{c+1}\left(1 + \frac{c}{(c+2)-(c+1)x}\right)$. Hence, $(g \circ f)^{(n)}(1) = n!\, c(c+1)^{n-1}$. This equality and (20) imply (19). ▮

Lemma 1.3. Let U be the operator (6) with $\tau = \tau(U) < 1$, $z \in \mathbb{C} \setminus \{0\}$ and $f \in C^\infty[u, v]$ be such that the function $g = Uf - zf$ is analytic. Then f is analytic and $r(f) \geqslant \min\{r(g), \gamma(U)\}$.

Proof. Pick arbitrary $R_a > 1/r(a)$ and $R_g > 1/r(g)$. Then according to (7) there exist $L_a, L_g \in (0, +\infty)$ such that

$$M_n(a) \leqslant L_a R_a^n \quad \text{and} \quad M_n(g) \leqslant L_g R_g^n \quad \text{for all} \quad n \in \mathbb{Z}_+. \tag{21}$$

Put $R_b = \frac{1}{\tilde{r}(b)}$, where $\tilde{r}(b)$ is defined in (7). According to (7) and (8)

$$M_n(b) \leqslant \tau R_b^{n-1} = \frac{\tau}{R_b} R_b^n \quad \text{for all} \quad n \in \mathbb{N}. \tag{22}$$

Pick now arbitrary $R > \max\{R_g, R_a, R_b/(1-\tau)\}$. Then

$$R > R_a, \quad R > R_g \quad \text{and} \quad \tau R + R_b < R. \tag{23}$$

Since $\tau^*(U) < 1$ there exists a positive integer j for which

$$M_j = \max_{k \geqslant j} \left| \int_Y a(x, y) \left(\frac{\partial b}{\partial x}(x, y) \right)^m \mu(dy) \right| < |z|.$$

Therefore there exists $q \in \mathbb{N}$, $q \geqslant j$ such that for all $l \geqslant q$,

$$\frac{L_g}{L_a \mu(Y)} R_g^l + R_a^l < R^l, \qquad \frac{L_a \mu(Y)\, \tau R}{(|z| - M_j)(\tau R + R_b)}\, l R_a^l < R^l \qquad \text{and} \tag{24}$$

$$\frac{L_a \mu(Y)\, \tau R}{(|z| - M_j)(\tau R + R_b - R_a)} (\tau R + R_b)^l < R^l \qquad \text{if} \quad \tau R + R_b > R_a. \tag{25}$$

Hence there exists $L > 1$ such that

$$M_n(f) \leqslant L R^n \tag{26}$$

for $n = 0, 1, \ldots, q$. We shall prove inductively that the inequality (26) holds also for $n = q+1, q+2, \ldots$ Suppose that $m > q$ and (26) holds for all $n < m$. We have to verify (26) for $n = m$. For this goal we differentiate the equality

$$z f(x) + g(x) = \int_Y a(x, y)\, f(b(x, y))\, \mu(dy)$$

m times and use Leibniz formula[27] and Lemma 1.1:

$$\begin{aligned} z f^{(m)}(x) &- \int_Y a(x, y) \left(\frac{\partial b}{\partial x}(x, y) \right)^m f^{(m)}(b(x, y))\, \mu(dy) \\ &= -g^{(m)}(x) + \int_Y \left(\frac{\partial^m a}{\partial x^m}(x, y)\, f(b(x, y)) \right. \\ &\quad + \sum_{n=1}^{m} \sum_{\substack{n = k_1 + \cdots + n k_n \\ k_1 \neq m}} \frac{\partial^{m-n} a}{\partial x^{m-n}}(x, y) \frac{m!\, f^{(k_1 + \cdots + k_n)}(b(x, y))}{(m-n)!\, k_1! \cdots k_n! (1!)^{k_1} \cdots (n!)^{k_n}} \\ &\quad \left. \times \left(\frac{\partial b}{\partial x}(x, y) \right)^{k_1} \cdots \left(\frac{\partial^n b}{\partial x^n}(x, y) \right)^{k_n} \right) \mu(dy). \end{aligned}$$

After obvious estimations using (7) we obtain

$$\begin{aligned} (|z| - M_j)\, M_m(f) & \\ \leqslant M_m(g) + \mu(Y) \Bigg(M_m(a)\, M_0(f) &+ \sum_{n=1}^{m} \sum_{\substack{n = k_1 + \cdots + n k_n \\ k_1 \neq m}} \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!} \\ \times M_{m-n}(a)\, M_{k_1 + \cdots + k_n}(f)\, M_1^{k_1}(b) \cdots M_n^{k_n}(b) \Bigg). & \end{aligned}$$

The induction hypothesis and inequalities (21), (22) imply that

$$M_m(f) \leqslant \frac{LL_a\mu(Y)}{|z|-M_j}\left(R_a^m + \frac{L_g R_g^m}{LL_a\mu(Y)} - R^m + \sum_{n=1}^{m} R_a^{m-n} R_b^n \sum_{n=k_1+\cdots+nk_n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!}\left(\frac{\tau R}{R_b}\right)^{k_1+\cdots+k_n}\right).$$

Since $L>1$ from (24) it follows that

$$M_m(f) \leqslant \frac{LL_a\mu(Y)}{|z|-M_j}\left(\sum_{n=1}^{m} R_a^{m-n} R_b^n \sum_{n=k_1+\cdots+nk_n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!}\left(\frac{\tau R}{R_b}\right)^{k_1+\cdots+k_n}\right). \tag{27}$$

According to Lemma 1.2

$$\sum_{n=k_1+\cdots+nk_n} \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!}\left(\frac{\tau R}{R_b}\right)^{k_1+\cdots+k_n} = \frac{\tau R}{R_b}\left(\frac{\tau R}{R_b}+1\right)^{n-1}. \tag{28}$$

Formulas (27) and (28) imply that

$$M_m(f) \leqslant \frac{LL_a\mu(Y)}{|z|-M_j}\,\frac{\tau R}{\tau R+R_b}\,R_a^m \sum_{n=1}^{m}\left(\frac{\tau R+R_b}{R_a}\right)^n. \tag{29}$$

Case 1. $\tau R+R_b \leqslant R_a$. In this case formulas (29) and (24) imply that

$$M_m(f) \leqslant \frac{LL_a\mu(Y)\,\tau R}{(|z|-M_j)(\tau R+R_b)}\,R_a^m m < LR^m.$$

Case 2. $\tau R+R_b > R_a$. In this case formulas (29), summation formula for geometric progression and inequality (25) imply that

$$M_m(f) \leqslant \frac{LL_a\mu(Y)\,\tau R((\tau R+R_b)^m - R_a^m)}{(|z|-M_j)(\tau R+R_b-R_a)} \leqslant \frac{LL_a\mu(Y)\,\tau R(\tau R+R_b)^m}{(|z|-M_j)(\tau R+R_b-R_a)} < LR^m.$$

Thus, in any case $M_m(f) \leqslant LR^m$, i.e., the inequality (26) is proved for all n. Therefore f is analytic and $r(f) \geqslant 1/R$. Since R is an arbitrary number greater then $\max\{R_g, R_a, R_b/(1-\tau)\}$, R_g is an arbitrary number greater then $1/r(g)$, R_a is an arbitrary number greater than $1/r(a)$ and $R_b = 1/\tilde{r}(b)$ we arrive to the iequality $r(f) \geqslant \min\{r(g), r(a), (1-\tau)\,\tilde{r}(b)\}$. ∎

Lemma 1.4. Let U be the operator (6) with $\tau(U)<1$, $k\in\mathbb{N}$, $z\in\mathbb{C}\setminus\{0\}$ and $f\in C^\infty[u,v]$ be such that $g=(U-zI)^k f$ is analytic. Then f is analytic and $r(f)\geqslant \min\{r(g),\gamma(U)\}$.

Proof. We shall use induction with respect to k. The case $k=1$ follows from Lemma 1.3. Let $k>1$ and suppose that the conclusion of the lemma is true for smaller k's. Since, $g=(U-zI)^{k-1}h$, where $h=Uf-zf$, the induction hypothesis implies that h is analytic and $r(h)\geqslant \min\{r(g),\gamma(U)\}$. Lemma 1.3 implies that f is analytic and $r(f)\geqslant \min\{r(h),\gamma(U)\}\geqslant \min\{r(g),\gamma(U)\}$. ∎

Now we shall prove Theorem 1. According to (9) for any $\varepsilon>0$, there exists $n\in\mathbb{N}$ such that $\gamma(U^n)>0$ and $\gamma(U^n)\geqslant\gamma^*(U)-\varepsilon$. Since $(U-zI)^k f=0$, we have that $(U^n-z^n)^k f=0$. Lemma 1.4 implies that f is analytic and $r(f)\geqslant\gamma(U^n)\geqslant\gamma^*(U)-\varepsilon$. Hence, $r(f)\geqslant\gamma^*(U)$.

4. PROOF OF THEOREM 2

Lemma 2.0. Let $k\in\mathbb{N}$, U_G be the operator (2), $z\in\mathbb{C}$, $|z|>R_k$ (see (14)) and $f\in C^k[0,1]$ be such that $U_G f-zf\in C^{k+1}[0,1]$. Then $f\in C^{k+1}[0,1]$.

Proof. Let $h=zf-U_G f$. Then $h\in C^{k+1}[0,1]$. Differentiating the equality $zf-U_G f=h$ k times with respect to x and using Lemma 1.1 and Leibnitz formula[27] we obtain that

$$g=zf^{(k)}-Wf^{(k)}\in C^1[0,1],\qquad \text{where}$$

$$W\varphi(x)=\sum_{n=1}^{\infty} a_n(x)(b_n'(x))^m\,\varphi(b_n(x)).$$

According to Theorem 2.5 of ref. 10, the spectral radius of the operator $W\colon C^1[0,1]\to C^1[0,1]$ does not exceed R_k. Since $|z|>R_k$, the operator $(zI-W)\colon C^1[0,1]\to C^1[0,1]$ is invertible. Since $(zI-W)\,f^{(k)}=g\in C^1[0,1]$, we obtain that $f^{(k)}=(zI-W)^{-1}g\in C^1[0,1]$. Therefore $f\in C^{k+1}[0,1]$. ∎

Lemma 2.1. Let $m,k\in\mathbb{N}$, U_G be the operator (2), $z\in\mathbb{C}$, $|z|>R_k$ (see (14)) and $f\in C^k[0,1]$ be such that $(U_G-zI)^m f\in C^{k+1}[0,1]$. Then $f\in C^{k+1}[0,1]$.

Proof. The case $m=1$ of Lemma 2.1 follows from Lemma 2.0. Let $m>1$ and suppose that for smaller m's Lemma 2.1 is already proved. Since

$(U_G - zI)^{m-1} (U_G - zI) f \in C^{k+1}[0,1]$, the induction hypothesis implies that $(U_G - zI) f \in C^{k+1}[0,1]$. Then $f \in C^{k+1}[0,1]$ according to Lemma 2.0. ▮

Lemma 2.2. Let U_G be the operator (2), $z \in \mathbb{C} \setminus \{0\}$, f be a function holomorphic in the 1-neighborhood of $[0,1]$ in $\mathbb{C}$ and $g \in \mathbf{H}$ be such that $U_G f(w) - z f(w) = g(w)$ for all $w \in [0,1]$. Then $f \in \mathbf{H}$, i.e., f admits an analytic extension to $\mathbb{C} \setminus (-\infty, -1]$ and this extension belongs to $\mathbf{H}$.

Proof. First, let us verify the following statement:

(A) If $[0,1] \subset W_0 \subset W_1 \subseteq \mathbb{C} \setminus (-\infty, -1]$, where W_0, W_1 are connected open subsets of $\mathbb{C}$ such that f admits an analytic extension to W_0 and $1/(w+n) \in W_0$ for any $n \in \mathbb{N}$ and any $w \in W_1$, then f admits an analytic extension to W_1.

Indeed, consider the function

$$h \colon W_1 \to \mathbb{C}, \qquad h(w) = \frac{1}{z}\left(-g(w) + \sum_{n=1}^{\infty} a_n(w)\, f(b_n(w)) \right), \tag{30}$$

where a_n and b_n are defined in (3). According to the above conditions and Weierstrass theorem[28] the function h is well-defined and analytic. Since $U_G f(w) - z f(w) = g(w)$ for all $w \in [0,1]$, formulas (2) and (30) imply that $h(w) = f(w)$ for all $w \in [0,1]$. According to the uniquness theorem,[28] h is the desired analytic extension of f. The statement (A) is proved.

Let W be the 1-neighborhood of $[0,1]$ in $\mathbb{C}$. (A) for $W_0 = W$ and $W_1 = W \cup \{w \in \mathbb{C} : \operatorname{Re} w > -1/2\}$ implies the existence of an analytic extension of f to $W \cup \{w \in \mathbb{C} : \operatorname{Re} w > -1/2\}$. Applying (A) to $W_0 = W \cup \{w \in \mathbb{C} : \operatorname{Re} w > -1/2\}$ and $W_1 = \{w \in \mathbb{C} : \operatorname{Re} w > -1\}$ we see that f admits an analytic extension to $\{w \in \mathbb{C} : \operatorname{Re} w > -1\}$. Applying (A) to $W_0 = \{w \in \mathbb{C} : \operatorname{Re} w > -1\}$ and $W_1 = \Omega_0 = \mathbb{C} \setminus \bigcup_{n=1}^{\infty} D_n$, where $D_n = \{w \in \mathbb{C} : |w - n - 1/2| \leqslant 1/2\}$, we obtain that f admits an analytic extension to Ω_0. Let now Ω_n $(n \in \mathbb{N})$ be sets

$$\Omega_n = \Omega_{n-1} \cup \{w \in \mathbb{C} : 1/(w+k) \in \Omega_{n-1} \text{ for any } k \in \mathbb{N}\}.$$

It is easy to see that Ω_n is an increasing sequence of open connected subsets of $\mathbb{C} \setminus (-\infty, -1]$. The statement (A) implies inductively the existence of an analytic extension of f to Ω_n for any $n \in \mathbb{N}$. Therefore f admits an analytic extension to $\Omega = \bigcup_{n=0}^{\infty} \Omega_n$. In order to prove the existence of an analytic extension of f to $\mathbb{C} \setminus (-\infty, -1]$ we have to show that $\Omega = \mathbb{C} \setminus (-\infty, -1]$. Let $w \in \mathbb{C} \setminus (\Omega_0 \cup (-\infty, -1])$. It suffices to verify that $w \in \Omega$. For this goal we consider the sequence: $w_0 = w$, $w_{k+1} = 1/(w_k + j_k)$, where $j_k = j(w_k)$ is the nearest natural number to $-w_k$ (this number is unique since $w_k \in \mathbb{C}\setminus(\Omega_0 \cup (-\infty, -1]))$. From the definition of Ω it follows that the inclusion $w \in \Omega$ will be proved if we demonstrate the existence of $k \in \mathbb{N}$ for which $w_k \in \Omega_0$. Suppose that such k does not exist. Then $w_k \notin \Omega_0$ for any k. Let $\alpha_k = \operatorname{Re} w_k$ and $\beta_k = \operatorname{Im} w_k$. Since $w_k \notin \Omega_0$ and $w_k \notin (-\infty, -1]$, we have that $\beta_k \neq 0$. By definition $w_{k+1} = 1/((\alpha_k + j_k) + \beta_k i) = (\alpha_k + j_k - \beta_k i)/((\alpha_k + j_k)^2 + \beta_k^2)$. Since $w_k \notin \Omega_0$ we obtain that $((\alpha_k + j_k)^2 + \beta_k^2) \leqslant 1/4$. Therefore $|\operatorname{Im} w_{k+1}| \geqslant 4\,|\beta_k| = 4\,|\operatorname{Im} w_k|$. Hence, $|\operatorname{Im} w_k| \to +\infty$, which contradicts the assumption $w_k \notin \Omega_0$. The existence of the desired analytic extension is proved.

It remains to show that the extended function f is bounded on each D_ε of (15). The uniqueness theorem implies that the equality $zf(w) = -g(w) + U_G f(w)$ is valid for any $w \in \mathbb{C}\setminus(-\infty, -1]$. Since the closure K of the set $\bigcup_{n=1}^{\infty} \{b_n(w) \colon w \in D_\varepsilon\}$ is a compact subset of $\mathbb{C}\setminus(-\infty, -1]$, we have that there exists $C_1 = C_1(\varepsilon) > 0$ such that $|g(w)| < C_1$ and $|f(b_n(w))| \leqslant C_1$ for any $n \in \mathbb{N}$ and $w \in D_\varepsilon$. On the other hand one can easily verify that there exists $C_2 = C_2(\varepsilon) > 0$ such that $\sum |a_n(w)| \leqslant C_2$ for any $w \in D_\varepsilon$. Then according to the equality $zf(w) = -g(w) + U_G f(w)$ we obtain that $|f(w)| \leqslant C_1(1 + C_2)/|z|$ for any $w \in D_\varepsilon$. Hence $f \in \mathbf{H}$. ∎

Lemma 2.3. Let $k \in \mathbb{N}$, U_G be the operator (2), $z \in \mathbb{C}\setminus\{0\}$, f be a function holomorphic in the 1-neighborhood of $[0, 1]$ in $\mathbb{C}$ and $g \in \mathbf{H}$ be such that $(U_G - zI)^k f(w) = g(w)$ for all $w \in [0, 1]$. Then $f \in \mathbf{H}$.

Proof. The case $k = 1$ follows from Lemma 2.2. Let $k > 1$ and suppose that the conclusion of the lemma is true for smaller k's. Since, $g = (U_G - zI)^{k-1} h$, where $h = Uf - zf$, the induction hypothesis implies that $h \in \mathbf{H}$. Then Lemma 2.2 implies that $f \in \mathbf{H}$. ∎

Now we shall prove Theorem 2. It is easy to see that the functions $|b'_{\mathbf{n}}(x)|$ decrease with respect to x and to any n_j, where $b_{\mathbf{n}}$ are functions defined in (10). Therefore

$$\tau_k = \sup\{|b'_{\mathbf{n}}(x)| : x \in [0, 1], \mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k\} = |b'_{\mathbf{1}}(0)|, \tag{31}$$

where $\mathbf{1} = (1, \dots, 1)$. Calculating the derivative of the rational function $b_{\mathbf{1}}$ we obtain

$$\tau_k = \prod_{j=1}^{k} \alpha_j^2, \qquad \text{where} \quad \alpha_1 = 1, \quad \alpha_{j+1} = 1/(1 + \alpha_j). \tag{32}$$

Since $\lim_{n\to\infty} \alpha_n^2 = 4/(\sqrt{5}+1)^2 = c$. Formulas (31), (32), (13), and (14) imply that

$$R_{m,k} \leqslant \tau_k^m = \prod_{j=1}^{k} \alpha_j^{2m} \Rightarrow R_k = \lim_{m\to\infty} (R_{m,k})^{1/m} \leqslant c^k.$$

Let now $z \in S_k$ and $f \in \mathscr{E}(z)$. According to Lemma 2.1 $f \in C^{k+1}[0,1]$. Using induction with respect to k, we see that $f \in C^\infty[0,1]$. In particular, the space $\mathscr{E}(z)$ does not depend on k provided $|z| > R_k$. Let us prove now that f admits an analytic extension to an 1-neighborhood of the segment $[0,1]$ in $\mathbb{C}$. For any $m \in \mathbb{N}$ the operator U_G^m has the form (6) with $Y = \mathbb{N}^m$ and $\mu(\mathbf{n}) = n_1^{-2} \cdot \cdots \cdot n_m^{-2}$. Clearly the parameter $\tau(U_G^m)$ defined in (8) is equal to τ_m of (31). According to (32) $\tau_m < 1$ for any $m \geqslant 2$. Therefore, we can apply Theorem 1. Calculations similar to the above calculation of τ_m show that the parameter $\tilde{r}(b)$ for U_G^m (see (7)) is equal to $1/\alpha_m$. Obviously the parameter $r(a)$ for U_G^m is equal to 1. According to Theorem 1 and formula (8)

$$\begin{aligned} r(f) &\geqslant \gamma^*(U) \geqslant \sup_{m\geqslant 2} \left(1 - \prod_{j=1}^{m} \alpha_j^2\right) \frac{1}{\alpha_m} \\ &\geqslant \lim_{m\to\infty} \left(1 - \prod_{j=1}^{m} \alpha_j^2\right) \frac{1}{\alpha_m} = \left(1 - \frac{4}{(\sqrt{5}+1)^2}\right) \frac{\sqrt{5}+1}{2} = 1. \end{aligned}$$

Definition of $r(f)$ implies that f admits an analytic extension to the 1-neighborhood of $[0,1]$. According to Lemma 2.3, $f \in \mathbf{H} \subset \mathscr{H}$. This proves that z-eigenspaces of the restrictions of U_G to $C^k[0,1]$, to $\mathbf{H}$ and to $\mathscr{H}$ are identical. Since the operator $U_G|_{\mathscr{H}}: \mathscr{H} \to \mathscr{H}$ is self-adjoint (Theorem MR), we obtain (16).

5. PROOF OF THEOREM 3

Case 1. $k = 1$. For any $f \in C^1[0,1]$ we have that $f = C + f_1$, where the constant C is defined by (5) and

$$f_1 \in E = \left\{ f \in C^1[0,1] : \int_0^1 \frac{f(x)}{x+1}\, dx = 0 \right\}.$$

Since $U_G 1 = 1$ and $U_G(E) \subseteq E$, we have $\langle U_G^n f, g\rangle = C\langle 1, g\rangle + \langle U_E^n f, g\rangle$, where $U_E = U_G|_E: E \to E$. According to Theorems 2 and MR, the spectral radius of U_E does not exceed c. Since $c < q$, we obtain that $|\langle U_G^n f, g\rangle - C\langle 1, g\rangle| = |\langle U_E^n f, g\rangle| \leqslant \|f\|\,\|g\|\,\|U_E^n\| = O(q^n)$.

Case 2. $k \geqslant 2$. Let L be the two-dimensional space spanned by the eigenvectors of U_G, corresponing to the eigenvalues $\lambda_0 = 1$ and λ_1. According to Theorems 2 and MR λ_0 and λ_1 are simple eigenvalues of the operator $U_G|_{C^k[0,1]}: C^k[0,1] \to C^k[0,1]$ (the essential spectral radius of this operator is less then $q = |\lambda_1|$). Then there exists a closed linear subspace M of $C^k[0,1]$ of codimension 2 such that $C^k[0,1] = L \oplus M$, and both closed linear subspaces L and M are invariant with respect to U_G. Let f_0 be the projection of f onto L along M and $f_1 = f - f_0$. Since the spectral radius of the operator $U_M = U_G|_M: M \to M$ is less than q, we have that $|\langle U_G^n f_1, g\rangle| \leqslant \|f_1\| \, \|g\| \, \|U_M^n\| = o(q^n)$. Standard arguments from linear algebra lead to $\langle U_G^n f_0, g\rangle = C + c\lambda_1^n$, where C is the constant defined by (5) and $c = c(f, g) \in \mathbb{C}$. Hence, $\langle U_G^n f, g\rangle = \langle U_G^n f_0, g\rangle + \langle U_G^n f_1, g\rangle = C + O(q^n)$.

6. CONCLUDING REMARKS

1. Let $z \neq 0$ and $f \in C^\infty[0,1]$ be a non-constant eigenfunction of the Frobenius–Perron operator U_G of the Gauss map. Theorem 2 implies that f admits an analytic extension to $\mathbb{C} \setminus (-\infty, -1]$. Analyzing the functional equation $U_G f = zf$ it is possible to show that the set of singularities of f is precisely the interval $(-\infty, -1]$. Moreover, for any $t \in (-\infty, -1]$ the analytic extension of f is unbounded in any upper half-disk and any lower half-disk with the center in t.

2. The class of integral operators (6) includes classical integral operators (for them $Y = [u, v]$ and $b(x, y) = y$), the evolution operators of dynamical systems and Markov processes. We may enlarge this class of operators replacing the interval $[u, v]$ by a compact analytic Riemannian manifold and considering such operators acting on vector-valued functions.

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